PARAMETRIC INSTABILITY OF GRAVITY WAVES ON THE SURFACE OF DEEP WATER

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We propose to investigate the characteristics of the parametric generation of gravity waves on the surface of a body of deep water. The threshold conditions for the onset of generation are determined, and the results are compared with the experimental data. The singularities of the excitation of parametric oscillations in a resonator are noted.

For surface gravity waves their resonance interaction is manifested in the third-order approximation in the parameter $\mu \sim (ka)$ (k is the wave number and a is the amplitude of the unperturbed wave) and satisfies the conditions

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4, \qquad \omega_1 + \omega_2 = \omega_3 + \omega_4 \tag{1}$$

The instability of a Stokes wave under small frequency and amplitude perturbations and the synchronous generation of a third harmonic [1, 2] are examples of this type of interaction that have been well studied. A special type of interactions satisfying conditions (1) is the parametric instability of a strong wave, as manifested in the self-excitation under definite conditions of a pair of waves having frequencies close to that of the primary (pump) wave. The cross scattering of waves by a standing pump wave has been observed experimentally both in the laboratory [3] and under natural conditions (near the shoreline of a bay) [4]. However, unlike second-order parametric instability (generation of subharmonic waves [3, 5]), the parametric instability of surface waves in a cubic-law medium have received very little attention. From the physical point of view this effect is analogous to the scattering of waves by waves in other nonlinear media, for example the scattering of light by light, which has been thoroughly investigated in nonlinear optics [6,7]. The indicated analogy permits us to expose the characteristic features of the third-order parametric instability of surface gravity waves.

We consider the case of degenerate parametric interaction, where

$$\begin{gathered} |\mathbf{k}_1| = |\mathbf{k}_2| = |\mathbf{k}_3| = |\mathbf{k}_4| = k \\ \omega_1 = \omega_2 = \omega_3 = \omega_4 \\ \mathbf{k}_1, \mathbf{k}_2 \perp \mathbf{k}_3, \mathbf{k}_4; \quad \mathbf{k}_1 \leftrightarrows \mathbf{k}_2; \quad \mathbf{k}_3 \rightleftarrows \mathbf{k}_4 \end{gathered} \tag{2}$$

A plane pump wave of frequency ω_1 generates a standing wave in the layer [y = 0, y = L]; the excited waves with frequencies $\omega_3 = \omega_4 = \omega$ propagate along the layer in mutually opposing directions.

The equations for the slowly varying amplitudes of the scattered waves are determined from the dynamic condition of constant pressure at the free surface of a deep liquid (z = 0); neglecting capillary effects, we can represent this condition in the form

$$\varphi_{ttz} - g(\varphi_{xx} + \varphi_{yy}) + {}^1/_2 g(\eta^2\varphi_{zzz})_z - g(\eta\varphi_{xz}\eta_x)_z - g(\eta\eta_y\varphi_{yz})_z + {}^1/_2(\eta^2\varphi_{tzz})_{tz} + (\eta\varphi_x\varphi_{xz})_{tz} + (\eta\varphi_z\varphi_{zz})_{tz} + (\eta\varphi_y\varphi_{yz})_{tz} = 0 \tag{3}$$

where $\varphi(x, y, z, t)$ is the hydrodynamic potential, $z = \eta(x, y, t)$ is the shape of the liquid surface, and g is the acceleration of gravity. Writing the solution of (3) in the form

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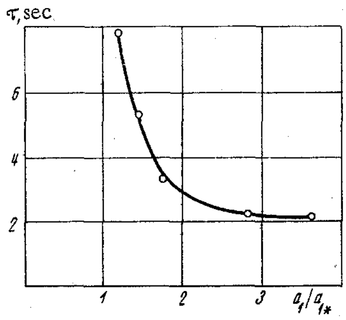

Fig. 1

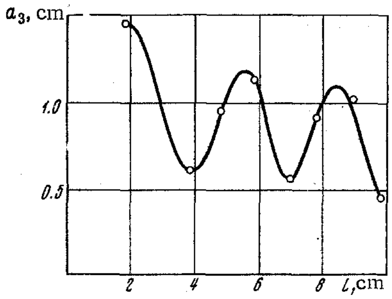

Fig. 2

$$\eta = \eta_1 + \eta_2 + \eta_3 = a_1 \cos ky e^{i\omega t} + a_3 e^{i(\omega t - kx)} + a_4 e^{i(\omega t + kx)} + (\ldots)^*$$

where $(\ldots)^*$ denotes the complex conjugate, and using the relationship between η and φ in the zeroth approximation, $\varphi = i\omega k^{-1}\eta$ at $z = 0$, we obtain

$$\frac{da_3}{dx} - \frac{1}{2} ik^3 a_4{}^* a_1{}^2 = 0, \qquad \frac{da_4}{dx} + \frac{1}{2} ik^3 a_3{}^* a_1{}^2 = 0 \tag{4}$$

Specifying the radiation conditions at the ends of a layer of length l in the form

$$a_3(0) = a_4(l) = 0, \quad a_3(l) = a_4(0) = a_0 \tag{5}$$

we obtain the solution (4) in the given pumping approximation:

$$a_3 = a_0 \sin(\sigma a_1{}^2 x), \quad a_4 = a_0 \cos(\sigma a_1{}^2 x) \tag{6}$$
$$(\sigma = {}^1/_2 k^3, \ k = \omega^2/g)$$

These amplitude distributions are meaningful only for generation near the threshold. The solution (6) makes it possible to determine the threshold pump amplitude, which is given by the expression

$$\sigma a_{1*}{}^2 l = \pi / 2 \tag{7}$$

The asterisk subscript denotes the threshold value of the pump amplitude.

If we take the attenuation of the scattered waves into account in the stage of the truncated equations [for which purpose it is sufficient to add terms $\sim \alpha a_{3,4}$ in Eq. (4)], the expression for the threshold pump amplitude acquires the form

$$a_{1*}{}^2 = \left\{ \left(\frac{\pi}{2\sigma l} \right)^2 + \left(\frac{\alpha}{\sigma} \right)^2 \right\}^{1/2} \tag{8}$$

For large l the threshold is determined mainly by the loss factor α.

In the underexcited regime, the oscillating layer represents the amplifier of surface waves directed along the layer and having a frequency close to the pump frequency. We note that if the mass transport of liquid is eliminated, exact synchronism is always realized in the parametric generation described above, because the nonlinear corrections to the wave vectors $\mathbf{k}_3$ and $\mathbf{k}_4$, which are oppositely directed, cancel one another.

Parametric generation at the pump frequency was investigated experimentally in a resonator comprising a narrow duct with an oscillating wide wall (rippler). The width of the duct did not exceed one to four pump half waves. The apparatus is described in [3]. The oscillation frequency of the rippler was set equal to 5.6 Hz. The depth of the water in the duct was h ~ 10 cm. The pump amplitude was varied by varying the stroke of the rippler, whose oscillations at the level of the liquid surface had a peak-to-peak amplitude in the range from 0 to 0.5 cm.

The wave amplitudes were measured by an optical method. The He–Ne laser beam reflected from the liquid surface during generation in the duct generally described an ellipse on the screen. By measuring the parameters of the ellipse (lengths of the semiaxes and their slopes) it was possible to determine the amplitudes and phases of the interacting waves.

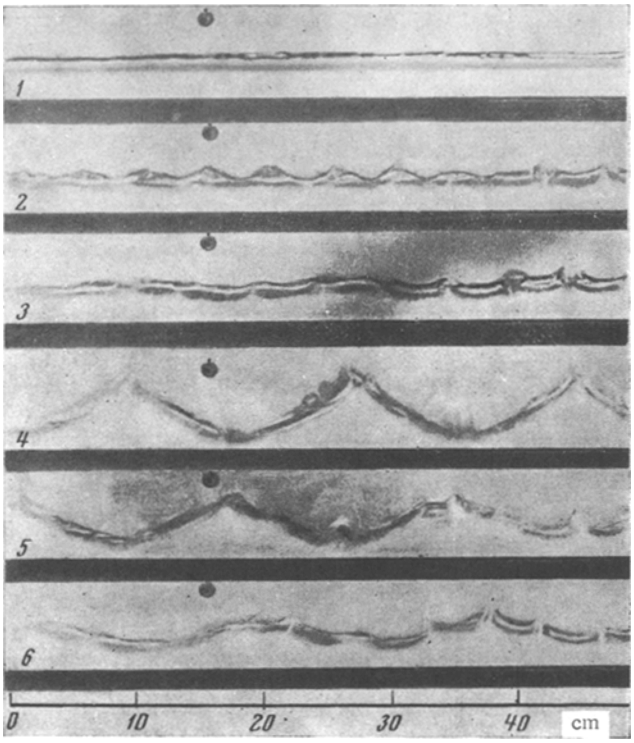

Fig. 3

The parametric generation of a wave of frequency ω was observed both under resonance and under nonresonance conditions (in the latter case smooth matching elements were used at the edges of the duct). The scattered waves appear as a cross-wave structure directed along the antinodal front of the pump wave, the oscillations at two adjacent antinodes differing in phase by π. This result indicates that the phases of the scattered waves can have values of either 0 or π relative to the pump phase.

The nonresonance case is characterized by the existence, for a fixed length of the layer, of a threshold pump amplitude at which generation is initiated [Eq. (7)]. A comparison of the theoretical values a_{1*} (for $\alpha = 0$) and the experimental values a'_{1*} of the threshold pump amplitude for various values of l is given below:

l, cm	20	30	40
a_{1*}, cm	0.28	0.22	0.20
a'_{1*}, cm	0.30	0.24	0.19

The cross-wave structure represents the superposition of two countertraveling waves, whose amplitudes vary smoothly from zero at one end of the duct to maximum values at the other end.

We measured the parametric generation buildup time as a function of the pump amplitude (Fig. 1). As the graph indicates, the buildup time τ decreases as a_1 increases, tending to a limiting value $\tau \approx 2$ sec, which is close to the single-transit time of the scattered wave over the layer (l = 50 cm). With a change of the layer width (l = const) the maximum values of the amplitudes a_3 and a_4 occur at resonance of the primary wave (Fig. 2).

The experimental results well support the above-described mechanism of wave scattering.

The parametric excitation of oscillations in a resonator has certain singular features. As shown by the experiments on the excitation of cross waves, three oscillation modes are possible in a rectangular resonator. For large depths ($h \gtrsim 5$ cm) stable subharmonic oscillations are established. The oscillation amplitude exceeds the pump amplitude, indicating that oscillations of the median level of the water in the resonator are responsible for this generation mode. As a rule, the appearance of a standing wave at the frequency $\omega/2$ is preceded by a cross-wave structure at the frequency ω, which builds up more rapidly than the subharmonic mode.

In a certain interval ($h \approx 4$ to 5 cm) the subharmonic oscillations are unstable, and energy is transferred back and forth periodically between the modes. The dynamics of this process is illustrated in the photographs of Fig. 3, which show the form of the surface oscillations in the resonator. Photograph 1 was taken 1 sec after activation of the rippler, and photographs 2 through 6 were taken in consecutive 5-sec intervals. For shallow depths stable scattering at the pump frequency is predominant. The increase in the parametric generation threshold at the subharmonic frequency with a decrease of h is apparently due to a redistribution of the pump energy in the higher harmonics. Even in a narrow duct the form of the standing wave already departs appreciably from sinusoidal, and scattering is observed at nonmultiple frequencies, primary at $3\omega/2$.

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LITERATURE CITED

1. O. M. Phillips, "Theoretical and experimental studies of gravity wave interactions," Proc. Roy. Soc., A299, No. 1456, 104-119 (1967).
2. V. E. Zakharov, "Stability of periodic finite-amplitude waves on the surface of deep water," Zh. Prikl. Mekhan. i Tekh. Fiz., No. 2 (1968).

3. A. S. Groshkov, V. F. Marchenko, and A. F. Tselykovskii, "Parametric wave generation on the surface of a liquid," Zh. Tekh. Fiz., No. 6 (1970).
4. A. I. Bowen and D. L. Inman, "Rip currents. 2. Laboratory and field observations," J. Geophys. Res., 74, No. 23 (1969).
5. C. J. R. Garret, "On cross waves," J. Fluid Mech., 41, Part 4 (1970).
6. A. P. Veduta and B. P. Kirsanov, "Stimulated quadrupole parametric scattering," Zh. Éksperim. i Teor Fiz., 56, No. 4 (1969).
7. D. N. Klyshko and N. I. Nazarova, "Scattering of light by light in a noncentrosymmetrical medium," Zh. Éksperim. i Teor. Fiz., 58, No. 3 (1970).
8. A. S. Gorshkov, V. F. Marchenko, and A. F. Tselykovskii, "Parametric amplification of waves on the surface of a liquid," Izv. VUZ, Radiofizika, 14, No. 2 (1971).